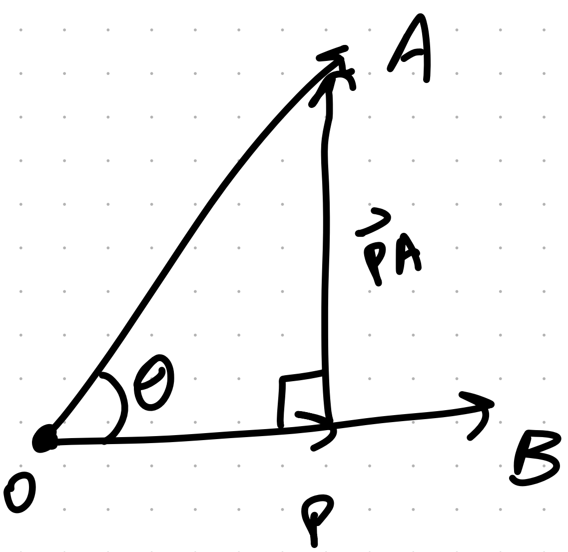
MA110 : Note on Component of Projection.

Consider vectors $A, B \in \mathbb{R}^n$

In Projection, we seek vector P

such that $P \parallel B$ & $P \perp \overrightarrow{PA}$.

Geometrically, we get the following picture:



To find P, we can go about two ways

① Let $P = cB$. This is the method detailed in the textbook.

Clearly, P is $\parallel$ to B.

Now we need to find c.

Consider our other constraint,

$$P \perp \overrightarrow{PA} \iff P \cdot (A - P) = 0$$

$$P \cdot A - P \cdot P = 0$$

$$c(A \cdot B) - c^2(B \cdot B) = 0$$

$$c = \frac{A \cdot B}{B \cdot B}$$

So we define $\boxed{\text{comp}_B(A) = \frac{A \cdot B}{B \cdot B}}$

and $\text{proj}_B(A) = \text{comp}_B(A)\, B$

$$\boxed{\text{proj}_B(A) = \left(\frac{A \cdot B}{B \cdot B}\right) B}$$

2. Alternatively, let $P = c\frac{B}{\|B\|}$,

i.e. we're now interpreting $c = \|P\|$.

We can perform a similar derivation to 1,

$$P \perp \overrightarrow{PA} \iff P \cdot (A - P) = 0$$

$$\frac{c}{\|B\|}(A \cdot B) - \frac{c^2}{\|B\|^2} B \cdot B = 0$$

$$C = \frac{A \cdot B}{\|B\|}$$

So we define $\boxed{\text{comp}_B(A) = \frac{A \cdot B}{\|B\|}}$

and $$\text{proj}_B(A) = \text{comp}_B(A) \frac{B}{\|B\|}$$

$$= \left(\frac{A \cdot B}{\|B\|^2}\right) B$$

$$\boxed{\text{proj}_B(A) = \frac{A \cdot B}{B \cdot B} B}$$

In sum, we can have two different definitions of "$\text{comp}_B(A)$" but they correspond to the same projection.

The $\text{comp}_B(A)$ formula in (2) is nice as it has a geometric interpretation as the length of P.

However, the $\text{comp}_B(A)$ formula in (2) yields the same result for <u>any</u>

non-zero vector parallel to B.
(but not anti parallel).

In this sense, we may refer to formula ② as the "component of A along B" and ① as the "component of A with respect to B" as formula ① varies w/ the length of B.

To see this, we rewrite ② in terms of the angle between A, B (θ)

$$\mathrm{comp}_B(A) = \frac{A \cdot B}{\|B\|} = \frac{\|A\|\|B\|\cos\theta}{\|B\|}$$

$$\boxed{\mathrm{comp}_B(A) = \|A\| \cos\theta}$$